THE SET OF COMMON FIXED POINTS OF AN n-PARAMETER CONTINUOUS SEMIGROUP OF MAPPINGS

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ABSTRACT. In this paper, using Kronecker's theorem, we discuss the set of common fixed points of an n-parameter continuous semigroup $\{T(p) : p \in \mathbb{R}^n_+\}$ of mappings. We also discuss convergence theorems to a common fixed point of an n-parameter nonexpansive semigroup $\{T(p) : p \in \mathbb{R}^n_+\}$.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the sets of all positive integers, all integers, all rational numbers and all real numbers, respectively. We put $\mathbb{R}^n_+ = [0, \infty)^n$ and

$$e_j = (0, 0, \cdots, 0, 0, \overset{(j)}{1}, 0, 0, \cdots, 0) \in \mathbb{R}^n$$

for $j \in \mathbb{N}$ with $1 \le j \le n$.

Let C be a subset of a Banach space E, and let T be a nonexpansive mapping on C, i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We know that T has a fixed point in the case that E is uniformly convex and C is bounded, closed and convex; see [4, 9]. See also [2, 3, 13] and others. We denote by $F(T)$ the set of fixed points of T.

Let τ be a Hausdorff topology on E. A family of mappings $\{T(p) : p \in \mathbb{R}^n_+\}$ is called an n-parameter τ-continuous semigroup of mappings on C if the following are satisfied:

(sg 1) $T(p + q) = T(p) \circ T(q)$ for all $p, q \in \mathbb{R}^n_+$;
(sg 2) for each $x \in C$, the mapping $p \mapsto T(p)x$ from $\mathbb{R}^n_+$ into C is continuous with respect to τ.

As a topology τ, we usually consider the strong topology of E. Also, a family of mappings $\{T(p) : p \in \mathbb{R}^n_+\}$ is called an n-parameter τ-continuous semigroup of nonexpansive mappings on C (in short, an n-parameter nonexpansive semigroup) if (sg 1), (sg 2) and the following (sg 3) are satisfied:

(sg 3) for each $p \in \mathbb{R}^n_+$, $T(p)$ is a nonexpansive mapping on C.

We know that an n-parameter nonexpansive semigroup $\{T(p) : p \in \mathbb{R}^n_+\}$ has a common fixed point in the case that E is uniformly convex and C is bounded, closed and convex; see Browder [4]. Moreover, in 1974, Bruck [7] proved that an n-parameter nonexpansive semigroup $\{T(p) : p \in \mathbb{R}^n_+\}$ has a common fixed point in the case that C is weakly compact, convex, and has the fixed point property for nonexpansive mappings.

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Very recently, the author proved the following in [16].

Theorem 1 ([16]). *Let E be a Banach space and let τ be a Hausdorff topology on E. Let $\{T(t) : t \geq 0\}$ be a 1-parameter τ-continuous semigroup of mappings on a subset C of E. Let α and β be positive real numbers satisfying $\alpha/\beta \notin \mathbb{Q}$. Then*

$$\bigcap_{t \geq 0} F\big(T(t)\big) = F\big(T(\alpha)\big) \cap F\big(T(\beta)\big)$$

holds.

Using this theorem, for an n-parameter τ-continuous semigroup $\{T(p) : p \in \mathbb{R}_+^n\}$ of mappings, we obtain

$$\bigcap_{p \in \mathbb{R}_+^n} F\big(T(p)\big) = \bigcap_{k=1}^{n} \Big(F\big(T(e_k)\big) \cap F\big(T(\sqrt{2} e_k)\big) \Big).$$

That is, the set of common fixed points of $\{T(p) : p \in \mathbb{R}_+^n\}$ is the set of common fixed points of $2n$ mappings.

In this paper, motivated by the above thing, we prove the direct generalization of Theorem 1 which says that the set of common fixed points of $\{T(p) : p \in \mathbb{R}_+^n\}$ is the set of common fixed points of some $n+1$ mappings. To prove it, we use Kronecker's theorem (Theorem 2). We also discuss convergence theorems to a common fixed point of n-parameter nonexpansive semigroups $\{T(p) : p \in \mathbb{R}_+^n\}$.

2. Preliminaries

In this section, we give some preliminaries. For a real number t, we denote by $[t]$ the maximum integer not exceeding t. It is obvious that $0 \leq t - [t] < 1$ for all $t \in \mathbb{R}$.

We use two kinds of the notions of linearly independent in this paper. We recall that vectors $\{p_1, p_2, \cdots, p_n\}$ is linearly independent in the usual sense if and only if there exist no $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n$ such that

$$(\lambda_1, \lambda_2, \cdots, \lambda_n) \neq 0 \quad \text{and} \quad \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n = 0.$$

On the other hand, we call that real numbers $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is linearly independent over $\mathbb{Q}$ if and only if there exist no $(\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{Z}^n$ such that

$$(\nu_1, \nu_2, \cdots, \nu_n) \neq 0 \quad \text{and} \quad \nu_1 \alpha_1 + \nu_2 \alpha_2 + \cdots + \nu_n \alpha_n = 0.$$

For example,

$$\Big\{1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13}, \sqrt{17}, \sqrt{19}, \sqrt{23}\Big\}$$

is linearly independent over $\mathbb{Q}$. For each irrational number γ, $\{1, \gamma\}$ is also linearly independent over $\mathbb{Q}$. The following theorem is Kronecker's theorem; see [11] and others.

Theorem 2 (Kronecker, 1884). *Let $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ such that $\{1, \alpha_1, \alpha_2, \cdots, \alpha_n\}$ is linearly independent over $\mathbb{Q}$. Then the set of cluster points of the sequence*

$$\Big\{ \Big(k\alpha_1 - [k\alpha_1],\ k\alpha_2 - [k\alpha_2],\ \cdots,\ k\alpha_n - [k\alpha_n] \Big) : k \in \mathbb{N} \Big\}$$

is $[0,1]^n$.

Let E be a Banach space. We recall that E is called strictly convex if $\|x+y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. E is called uniformly convex if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x + y\|/2 < 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. It is obvious that a uniformly convex Banach space is strictly convex. The norm of E is called Fréchet differentiable if for each $x \in E$ with $\|x\| = 1$, $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists and is attained uniformly in $y \in E$ with $\|y\| = 1$. The following lemma is the corollary of Bruck's result in [6].

Lemma 1 (Bruck [6])**.** *Let C be a subset of a strictly convex Banach space E. Let $\{T_1, T_2, \cdots, T_\ell\}$ be a family of nonexpansive mappings from C into E with a common fixed point. Let $\lambda_1, \lambda_2, \cdots, \lambda_\ell \in (0, 1]$ such that $\sum_{j=1}^{\ell} \lambda_j = 1$. Then a mapping S from C into E defined by*

$$Sx = \lambda_1 T_1 x + \lambda_2 T_2 x + \cdots + \lambda_\ell T_\ell x$$

for $x \in C$ is nonexpansive and

$$F(S) = F(T_1) \cap F(T_2) \cap \cdots \cap F(T_\ell)$$

holds.

3. Main Results

In this section, we prove our main results.

Theorem 3. *Let E be a Banach space and let τ be a Hausdorff topology on E. Let $\{T(p) : p \in \mathbb{R}^n_+\}$ be an n-parameter τ-continuous semigroup of mappings on a subset C of E. Let $p_1, p_2, \cdots, p_n \in \mathbb{R}^n_+$ such that $\{p_1, p_2, \cdots, p_n\}$ is linearly independent in the usual sense. Let $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ such that $\{1, \alpha_1, \alpha_2, \cdots, \alpha_n\}$ is linearly independent over $\mathbb{Q}$, and*

$$p_0 = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n \in \mathbb{R}^n_+.$$

Then

$$\bigcap_{p\in\mathbb{R}^n_+} F\big(T(p)\big) = F\big(T(p_0)\big) \cap F\big(T(p_1)\big) \cap \cdots \cap F\big(T(p_n)\big)$$

holds.

To prove it, we need some lemmas. In the following lemmas and the proof of Theorem 3, we let

$$z \in F\big(T(p_0)\big) \cap F\big(T(p_1)\big) \cap \cdots \cap F\big(T(p_n)\big).$$

That is,

$$T(p_0)z = T(p_1)z = \cdots = T(p_n)z = z.$$

Also, we put

$$\ell = \max\Big\{\big[|\alpha_j|\big] + 1 : 1 \leq j \leq n\Big\} \in \mathbb{N}, \quad \beta_k = \alpha_k + \ell > 0$$

for $k \in \mathbb{N}$ with $1 \leq k \leq n$, and

$$p_0' = \beta_1 p_1 + \beta_2 p_2 + \cdots + \beta_n p_n \in \mathbb{R}^n_+.$$

Lemma 2. *$T(p_0')z = z$ holds.*

Proof. Since z is a common fixed point of $\{T(p_k) : k \in \mathbb{N}, 0 \le k \le n\}$, we have

$$\begin{aligned} T(p_0')z &= T(\beta_1 p_1 + \beta_2 p_2 + \cdots + \beta_n p_n)z \\ &= T(p_0 + \ell p_1 + \ell p_2 + \cdots + \ell p_n)z \\ &= T(p_0) \circ T(p_1)^\ell \circ T(p_2)^\ell \circ \cdots \circ T(p_n)^\ell z \\ &= z. \end{aligned}$$

This completes the proof. □

Lemma 3. *For every* $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in [0,1)^n$,

$$T(\lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n)z = z$$

holds.

Proof. We first show $\{1, \beta_1, \beta_2, \cdots, \beta_n\}$ is linearly independent over $\mathbb{Q}$. Assume that

$$\nu_0 + \nu_1 \beta_1 + \nu_2 \beta_2 + \cdots + \nu_n \beta_n = 0$$

for some $(\nu_0, \nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{Z}^n$. Then by the definition of β_j, we obtain

$$(\nu_0 + \nu_1 \ell + \nu_2 \ell + \cdots + \nu_n \ell) + \nu_1 \alpha_1 + \nu_2 \alpha_2 + \cdots + \nu_n \alpha_n = 0.$$

Since $\nu_0 + \nu_1 \ell + \nu_2 \ell + \cdots + \nu_n \ell \in \mathbb{Z}$ and $\{1, \alpha_1, \alpha_2, \cdots, \alpha_n\}$ is linearly independent over $\mathbb{Q}$, we have

$$\nu_0 + \nu_1 \ell + \nu_2 \ell + \cdots + \nu_n \ell = \nu_1 = \nu_2 = \cdots = \nu_n = 0.$$

From this, we also have $\nu_0 = 0$. Therefore $\{1, \beta_1, \beta_2, \cdots, \beta_n\}$ is linearly independent over $\mathbb{Q}$. So, by Kronecker's theorem (Theorem 2), there exists a sequence $\{\ell_k\}$ in $\mathbb{N}$ such that $\ell_k < \ell_{k+1}$ for $k \in \mathbb{N}$ and

$$\lim_{k \to \infty} \ell_k \beta_j - [\ell_k \beta_j] = \lambda_j$$

for all $j \in \mathbb{N}$ with $1 \le j \le n$. We next show

$$T\left(\sum_{j=1}^n \big(\ell_k \beta_j - [\ell_k \beta_j]\big) p_j\right) z = z$$

for all $k \in \mathbb{N}$. We define $T(p_j)^0$ is the identity mapping on C. For each $k \in \mathbb{N}$, we have

$$\begin{aligned} & T\left(\sum_{j=1}^n \big(\ell_k \beta_j - [\ell_k \beta_j]\big) p_j\right) z \\ &= T\left(\sum_{j=1}^n \big(\ell_k \beta_j - [\ell_k \beta_j]\big) p_j\right) \circ T(p_1)^{[\ell_k \beta_1]} \circ T(p_2)^{[\ell_k \beta_2]} \circ \cdots \circ T(p_n)^{[\ell_k \beta_n]} z \\ &= T\left(\sum_{j=1}^n \ell_k \beta_j p_j\right) z \\ &= T\left(\sum_{j=1}^n \beta_j p_j\right)^{\ell_k} z \\ &= T\left(p_0'\right)^{\ell_k} z = z \end{aligned}$$

by Lemma 2. Since

$$\lim_{k\to\infty}\sum_{j=1}^{n}\big(\ell_k\beta_j-[\ell_k\beta_j]\big)p_j=\sum_{j=1}^{n}\lambda_j p_j,$$

we obtain the desired result. □

Lemma 4. *For every* $(\lambda_1,\lambda_2,\cdots,\lambda_n)\in[0,\infty)^n$,

$$T(\lambda_1 p_1+\lambda_2 p_2+\cdots+\lambda_n p_n)z=z$$

holds.

Proof. By Lemma 3, we have

$$\begin{aligned}
&T\left(\sum_{j=1}^{n}\lambda_j p_j\right)z\\
&=T\left(\sum_{j=1}^{n}\big(\lambda_j-[\lambda_j]\big)p_j\right)\circ T(p_1)^{[\lambda_1]}\circ T(p_2)^{[\lambda_2]}\circ\cdots\circ T(p_n)^{[\lambda_n]}z\\
&=T\left(\sum_{j=1}^{n}\big(\lambda_j-[\lambda_j]\big)p_j\right)z\\
&=z.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 3. We fix $p\in\mathbb{R}_+^n$. Since $\{p_1,p_2,\cdots,p_n\}$ is linearly independent in the usual sense, there exists $(\lambda_1,\lambda_2,\cdots,\lambda_n)\in\mathbb{R}^n$ such that

$$p=\lambda_1 p_1+\lambda_2 p_2+\cdots+\lambda_n p_n.$$

Put

$$m=\max\Big\{\big[|\lambda_j|\big]+1:1\le j\le n\Big\}\in\mathbb{N}.$$

We note that $\lambda_j+m>0$ for all j. By Lemma 4, we obtain

$$\begin{aligned}
T(p)z&=T\left(\sum_{j=1}^{n}\lambda_j p_j\right)z\\
&=T\left(\sum_{j=1}^{n}\lambda_j p_j\right)\circ T(p_1)^m\circ T(p_2)^m\circ\cdots\circ T(p_n)^m z\\
&=T\left(\sum_{j=1}^{n}\big(\lambda_j+m\big)p_j\right)z\\
&=z.
\end{aligned}$$

Since $p\in\mathbb{R}_+^n$ is arbitrary, we obtain the desired result. □

Theorem 3 is the direct generalization of Theorem 1. We give the proof of Theorem 1 by using Theorem 3.

Proof of Theorem 1. Put $p_1 = \beta$. Since $p_1 \neq 0$, $\{p_1\}$ is linearly independent in the usual sense. Put $\alpha_1 = \alpha/\beta \in \mathbb{R} \setminus \mathbb{Q}$. We note that $\{1, \alpha_1\}$ is linearly independent over $\mathbb{Q}$. Put $p_0 = \alpha_1 p_1$. Then by Theorem 3, we obtain

$$\bigcap_{t \geq 0} F\big(T(t)\big) = F\big(T(p_0)\big) \cap F\big(T(p_1)\big) = F\big(T(\alpha)\big) \cap F\big(T(\beta)\big).$$

This completes the proof. □

As another direct consequence of Theorem 3, we obtain the following.

Corollary 1. *Let E be a Banach space and let τ be a Hausdorff topology on E. Let $\{T(p) : p \in \mathbb{R}_+^n\}$ be an n-parameter τ-continuous semigroup of mappings on a subset C of E. Put α_k the square root of the k-th prime number for $k \in \mathbb{N}$ with $1 \leq k \leq n$, and*

$$p_0 = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n \in \mathbb{R}_+^n.$$

Then

$$\bigcap_{p \in \mathbb{R}_+^n} F\big(T(p)\big) = F\big(T(p_0)\big) \cap F\big(T(e_1)\big) \cap F\big(T(e_2)\big) \cap \cdots \cap F\big(T(e_n)\big)$$

holds.

Using Lemma 1, we obtain the following.

Corollary 2. *Let E be a strictly convex Banach space and let τ be a Hausdorff topology on E. Let $\{T(p) : p \in \mathbb{R}_+^n\}$ be an n-parameter τ-continuous semigroup of nonexpansive mappings on a subset C of E with a common fixed point. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $\{p_0, p_1, p_2, \cdots, p_n\}$ as in Theorem 3. Define a nonexpansive mapping S from C into E by*

$$Sx = \lambda_0 T(p_0)x + \lambda_1 T(p_1)x + \cdots + \lambda_n T(p_n)x$$

for $x \in C$, where $\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_n \in (0,1)$ with $\sum_{j=0}^n \lambda_j = 1$. Then

$$\bigcap_{p \in \mathbb{R}_+^n} F\big(T(p)\big) = F(S)$$

holds.

Corollary 3. *Let E be a uniformly convex Banach space and let τ be a Hausdorff topology on E. Let $\{T(p) : p \in \mathbb{R}_+^n\}$ be an n-parameter τ-continuous semigroup of nonexpansive mappings on a bounded closed convex subset C of E. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $\{p_0, p_1, p_2, \cdots, p_n\}$ as in Theorem 3. Define a nonexpansive mapping S on C as Corollary 2. Then*

$$\bigcap_{p \in \mathbb{R}_+^n} F\big(T(p)\big) = F(S)$$

holds.

4. Convergence Theorems

Using Theorem 3, we can prove many convergence theorems to a common fixed point of an n-parameter τ-continuous semigroup $\{T(p) : p \in \mathbb{R}_+^n\}$ of nonexpansive mappings. In this section, we state some of them. In the following theorems, we always let E, τ, C, $\{T(p)\}$, $\{p_j\}$, $\{\alpha_j\}$ and $\{\lambda_j\}$ as follows:

- Let E be a Banach space and let τ be a Hausdorff topology on E. Let $\{T(p) : p \in \mathbb{R}_+^n\}$ be an n-parameter τ-continuous semigroup of nonexpansive mappings on a bounded closed convex subset C of E. Let $p_1, p_2, \cdots, p_n \in \mathbb{R}_+^n$ such that $\{p_1, p_2, \cdots, p_n\}$ is linearly independent in the usual sense. Let $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ such that $\{1, \alpha_1, \alpha_2, \cdots, \alpha_n\}$ is linearly independent over $\mathbb{Q}$, and $p_0 = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n \in \mathbb{R}_+^n$. Let $\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_n \in (0,1)$ such that $\sum_{j=0}^n \lambda_j = 1$.

By the results of Bruck [8] and Reich [14], we obtain the following; see also Baillon [1].

Theorem 4. *Assume that E is uniformly convex and the norm of E is Fréchet differentiable. Define a nonexpansive mapping S on C by*

$$Sx = \sum_{j=0}^n \lambda_j T(p_j)x$$

for all $x \in C$. Define two sequences $\{x_k\}$ and $\{y_k\}$ in C by

$$x \in C, \quad x_k = \frac{Sx + S^2x + S^3x + \cdots + S^kx}{k}$$

for $k \in \mathbb{N}$, and

$$y_1 \in C, \quad y_{k+1} = \frac{1}{2} S y_k + \frac{1}{2} y_k$$

for $k \in \mathbb{N}$. Then $\{x_k\}$ and $\{y_k\}$ converge weakly to a common fixed point of $\{T(p) : p \in \mathbb{R}_+^n\}$.

By the results of Browder [5] and Wittmann [17], we obtain the following; see also Halpern [10].

Theorem 5. *Assume that E is a Hilbert space. Define a nonexpansive mapping S on C as Theorem 4. Let $\{s_k\}$ and $\{t_k\}$ be sequences in $(0,1)$ satisfying*

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} t_k = 0, \quad \sum_{k=1}^\infty t_k = \infty, \quad \text{and} \quad \sum_{k=1}^\infty \mid t_{k+1} - t_k \mid < \infty.$$

Define two sequences $\{x_k\}$ and $\{y_k\}$ in C as

$$x_k = (1 - s_k) S x_k + s_k u$$

for $k \in \mathbb{N}$, and

$$y_1 \in C, \quad y_{n+1} = (1 - t_k)\, S y_k + t_k\, u$$

for $k \in \mathbb{N}$. Then $\{x_k\}$ and $\{y_k\}$ converges strongly to a common fixed point of $\{T(p) : p \in \mathbb{R}_+^n\}$.

By the result of Rodé [15], we obtain the following.

Theorem 6. *Assume that E is a Hilbert space. Define a sequence $\{x_k\}$ in C by*

$$x \in C \quad \text{and} \quad x_k = \frac{\sum \left\{ T\left(\sum_{j=0}^{n} \nu_j p_j\right) x : \nu_j \in \{1, 2, \cdots, k\} \right\}}{k^{n+1}}$$

for $k \in \mathbb{N}$. Then $\{x_k\}$ converges weakly to a common fixed point of $\{T(p) : p \in \mathbb{R}_+^n\}$.

By the result of Ishikawa [12], we obtain the following.

Theorem 7. *Assume that C is compact. Define mappings S_j on C by*

$$S_j x = \frac{1}{2} T(p_j) x + \frac{1}{2} x$$

for all $x \in C$ and $j = 0, 1, 2, \cdots, n$. Let $x_1 \in C$ and define a sequence $\{x_k\}$ in C by

$$x_{k+1} = \left[\prod_{k_n=1}^{k} \left[S_n \prod_{k_{n-1}=1}^{k_n} \left[S_{n-1} \cdots \left[S_2 \prod_{k_1=1}^{k_2} \left[S_1 \prod_{k_0=1}^{k_1} S_0 \right] \right] \cdots \right] \right] \right] x_1$$

for $n \in \mathbb{N}$. Then $\{x_k\}$ converges strongly to a common fixed point of $\{T(p) : p \in \mathbb{R}_+^n\}$.

5. Counterexample

In Corollary 2, we assume that $\{T(p) : p \in \mathbb{R}_+^n\}$ has a common fixed point. The following example says this assumption is needed.

Example 1. Put $E = C = \mathbb{R}$ and let τ be the usual topology on E. Define a 2-parameter τ-continuous semigroup $\{T(p) : p \in \mathbb{R}_+^2\}$ of nonexpansive mappings on C by

$$T(\lambda_1 e_1 + \lambda_2 e_2) x = x + \lambda_1 - \lambda_2$$

for $\lambda_1, \lambda_2 \in [0, \infty)$ and $x \in E$. Define a nonexpansive mapping S on C by

$$Sx = \frac{\sqrt{2} + \sqrt{3} + 1}{6} T(\sqrt{2} e_1 + \sqrt{3} e_2) x + \frac{3 - \sqrt{2}}{6} T(e_1) x + \frac{2 - \sqrt{3}}{6} T(e_2) x$$

for $x \in C$. Then

$$\bigcap_{p \in \mathbb{R}_+^n} F\big(T(p)\big) = \varnothing \subsetneqq C = F(S)$$

holds.

Proof. Since $F\big(T(e_1)\big) = \varnothing$ and $Sx = x$ for all $x \in C$, we obtain the desired result. □

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